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Random field O(N) spin model near four dimensions

Yoshinori Sakamoto[1], **Hisamitsu Mukaida**[2] **and Chigak Itoi**[3]

[1] Laboratory of Physics, College of Science and Technology, Nihon University, 7-24-1 Narashino-dai, Funabashi-city, Chiba 274-8501, Japan
[2] Department of Physics, Saitama Medical University, 981 Kawakado, Iruma-gun, Saitama 350-0496, Japan
[3] Department of Physics, College of Science and Technology, Nihon University, 1-8-14 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan

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Abstract
We study the stability of fixed points in the two-loop renormalization group for the random field O(N) spin model in $4+\epsilon$ dimensions. We obtain fixed points in the $1/N$ expansion and ϵ expansion. We solve the eigenvalue equation for an infinitesimal deviation from these fixed points under physical conditions on the random anisotropy function. We find that the fixed point corresponding to dimensional reduction is singly unstable for sufficiently large N, and the critical exponents show a dimensional reduction. Also, we derive the condition on N for the non-existence of the fixed point corresponding to dimensional reduction. The result is qualitatively consistent with that in recent papers.

The critical phenomena in the random field O(N) spin model is worth studying from the viewpoint of quenched disorder and spin correlations. Dimensional reduction [1] is one key to clarifying the nature of this model. Since several rigorous results for the random field Ising model ($N = 1$ case) indicated the failure of dimensional reduction to predict the lower critical dimensions [2–4], the breakdown of dimensional reduction with some approximation methods was discussed in order to obtain an intuitive understanding or quantitative information. Fisher calculated the one-loop renormalization group, and pointed out the breakdown of dimensional reduction due to the appearance of an infinite number of relevant operators in $4+\epsilon$ dimensions [5]. Feldman found a non-analytic fixed point in Fisher's renormalization group for several small N [6]. He obtained nontrivial critical exponents shifted from the predictions of dimensional reduction. Mézard and Young also suggested breakdown of dimensional reduction by replica symmetry breaking [7]. Now, many researchers believe that the dimensional reduction is incorrect in dimensions lower than six.

Recently, Tarjus and Tissier studied the critical phenomena of this model in any dimension and for any value of N by using the nonperturbative renormalization group method and the replica method [8]. They showed the following relation of the critical exponents of the two-point spin correlation function

$$\eta = \bar{\eta} = \frac{\epsilon}{N-2}, \tag{1}$$

predicted by dimensional reduction in a certain region in the (d, N) plane. Since this relation seems valid for $N \geqslant 18$ near four dimensions, the consistency between their result and that of the references [5, 7] should be studied.

To understand the consistency of their works, the de Almeida–Thouless criterion [9] is applied faithfully to this model in a simple $1/N$-expansion method [10]. It is shown that the saddle point of the auxiliary field is stable in the random field O(N) spin model. Also the stability argument by Balents and Fisher for random media [11] is applied to Fisher's one-loop renormalization group; then, the following two possibilities are indicated [10]. The fixed point corresponding to dimensional reduction is singly unstable, or there is no singly unstable fixed point. Combining these results leads to relation (1) which is the same as the result obtained by Tarjus and Tissier.

In this paper we study the stability of the fixed point corresponding to dimensional reduction by the two-loop renormalization group obtained by Le Doussal and Wiese [12] and Tissier and Tarjus [13]. A detailed analysis of all the fixed points and their stability in the large-N limit has been provided in our recent work [14]. For sufficiently large but finite N, the solution of the eigenvalue equation shows that the unstable modes pointed out by Fisher [5] are fictitious. Therefore, we conclude that the fixed point yielding the relation (1) is singly unstable for sufficiently large N. We also discuss the uniqueness of this fixed point.

We consider O(N) classical spins $\boldsymbol{S}(x)$ with a fixed-length constraint $\boldsymbol{S}(x)^2 = 1$. To take the average over the random field, one introduces replicas $\boldsymbol{S}^\alpha(x)$, $\alpha = 1, \ldots, n$. We start from a nonlinear σ model of the following replica partition function and action:

$$\mathcal{Z} = \int \prod_{\alpha=1}^{n} \mathcal{D}\boldsymbol{S}^\alpha \delta(\boldsymbol{S}^{\alpha 2} - 1) \mathrm{e}^{-\beta H_{\mathrm{rep}}},$$

$$\beta H_{\mathrm{rep}} = \frac{a^{2-d}}{2T} \int \mathrm{d}^d x \sum_{\alpha=1}^{n} \sum_{\mu=1}^{d} (\partial_\mu \boldsymbol{S}^\alpha)^2 - \frac{a^{-d}}{2T^2} \int \mathrm{d}^d x \sum_{\alpha,\beta}^{n} R(\boldsymbol{S}^\alpha \cdot \boldsymbol{S}^\beta), \tag{2}$$

where a is the ultraviolet cutoff and the parameter T denotes the dimensionless temperature. The function $R(\boldsymbol{S}^\alpha \cdot \boldsymbol{S}^\beta)$ represents general anisotropy including the random field and all the random anisotropies, and is given by

$$R(\boldsymbol{S}^\alpha \cdot \boldsymbol{S}^\beta) = \sum_{\mu=1}^{\infty} \Delta_\mu (\boldsymbol{S}^\alpha \cdot \boldsymbol{S}^\beta)^\mu, \tag{3}$$

where Δ_μ denotes the strength of the random field and the μth rank random anisotropy ($\mu = 1$ is the random field, and $\mu \geqslant 2$ is the second- and higher-rank random anisotropy). These coupling constants are positive semidefinite for $\Delta_\mu \geqslant 0$.

The beta function $\partial_t R(z)$ at zero temperature can be expressed in the loop expansion

$$\partial_t R(z) = \beta_0[R] + \beta_1[R] + \beta_2[R] + \cdots. \tag{4}$$

Here we have defined the scale parameter t which increases toward the infrared direction. Fisher calculated the one-loop beta function [5]. Recently, Le Doussal and Wiese calculated a two-loop beta function at zero temperature [12]. Independently, Tarjus and Tissier obtained a consistent result [13]. The two-loop beta function is given by

$$\beta_0[R] = -\epsilon R(z), \tag{5}$$

$$\begin{aligned} \beta_1[R] = {} & 2(N-2)R'(1)R(z) - (N-1)zR'(1)R'(z) + (1-z^2)R'(1)R''(z) \\ & + \tfrac{1}{2}R'(z)^2(N-2+z^2) - R'(z)R''(z)z(1-z^2) + \tfrac{1}{2}R''(z)^2(1-z^2)^2, \end{aligned} \tag{6}$$

$$\begin{aligned} \beta_2[R] = {} & \tfrac{1}{2}(N-2)[(1-z^2)^2R''(z)^3 - (1-z^2)(3zR'(z) - (2+z^2)R'(1))R''(z)^2 \\ & - 2(1-z^2)(R'(z) - zR'(1))R'(z)R''(z) + (1-z^2)R'(1)R'(z)^2 \end{aligned}$$

$$+ 4R'(1)^2R(z)] - \tfrac{1}{2}(1-z^2)[(1-z^2)R'''(z) - 3zR''(z) - R'(z)]^2$$
$$\times [-(1-z^2)R''(z) + zR'(z) - R'(1)]$$
$$- \frac{c^2}{2}[(N+2)(1-z^2)R''(z) - (3N-2)zR'(z) + 8K(N-2)R(z)], \tag{7}$$

where $c = \lim_{z\nearrow 1} \sqrt{1-z^2}R''(z)$ and $K = 2\gamma_a$ is an unknown real number.

The fixed-point condition of the renormalization group determines properties of the function $R(z)$. The possible singularity in the fixed-point function $R'(z) \sim (1-z)^\alpha$ and the possible deformation from the fixed point is given by $\alpha = \frac{1}{2}$ or $\alpha \geqslant 1$. Only in this case, the nontrivial critical behaviour differs from the prediction of the dimensional reduction. Since the initial value $R(z)$ of the renormalization group equation (4) is an analytic function, the flow of $R''(1)$ should diverge for the breakdown of the dimensional reduction.

The fixed point corresponding to the dimensional reduction has the correction $R'(1) = \frac{\epsilon}{N-2} - \left(\frac{\epsilon}{N-2}\right)^2$. The fixed-point equation for $R''(1)$ becomes a cubic algebraic equation, which has only one real solution for $N < 18 - \frac{49}{5}\epsilon$. In this case, the dimensional reduction breaks. This condition is identical to the existence condition for a suitable cuspy fixed point obtained by Le Doussal and Wiese [12], which is consistent also with the phase diagram obtained by Tarjus and Tissier.

The two-loop beta function (7) enables us to calculate the higher order corrections to the fixed point corresponding to the dimensional reduction. For $N \geqslant 18 - \frac{49}{5}\epsilon$, this fixed point may exist, and it can be obtained in a double expansion with respect to $1/N$ and ϵ to several orders. We expand the fixed point up to necessary orders to discuss its stability:

$$R(z) = \frac{\epsilon}{N}\left(z - \frac{1}{2}\right) + \frac{\epsilon}{N^2}\left(\frac{1}{2}z^2 + z\right) + \frac{\epsilon^2}{N^2}\left(-\frac{1}{2}z^2 - \frac{1}{2}\right) + \cdots. \tag{8}$$

We study the stability of this fixed point in the scaling operator. In the two-loop analysis, we define the deviation function from this fixed point by

$$\delta R'(z) = \frac{\epsilon}{N}v(z).$$

We can calculate the eigenvalue equation for the deviation

$$(1-\epsilon)(1-z)^2(1+z)v''(z) + [N - 4z - 2 + \epsilon(2+4z)]\,(1-z)v'(z)$$
$$+\ [2(1-\epsilon)z - N\lambda]\,v(z) + [N - 2 + \epsilon(4-z)]\,v(1) = 0. \tag{9}$$

First, we study the equation for $v(1) = 0$. The general solution is given by a linear combination of the two independent solutions:

$$v(z) = a_0^+ v(z, \alpha_+(\lambda)) + a_0^- v(z, \alpha_-(\lambda)), \tag{10}$$

where $v(z,\alpha)$ can be expressed in terms of the Gaussian hypergeometric function

$$v(z,\alpha) = (1-z)^\alpha\, {}_2F_1\left(1+\alpha,\ 2+\alpha;\ 3+2\alpha - \frac{N}{2(1-\epsilon)};\ \frac{1-z}{2}\right).$$

The generalized hypergeometric function is defined by the following series expansion:

$${}_mF_n\,(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n; z) \equiv \sum_{k=0}^{\infty} \frac{(x_1)_k(x_2)_k\cdots(x_m)_k}{(y_1)_k(y_2)_k\cdots(y_n)_k}\frac{z^k}{k!}.$$

$\alpha_\pm(\lambda)$ is defined by

$$\alpha = \alpha_\pm(\lambda) \equiv \frac{N - 4(1-\epsilon) \pm \sqrt{N^2 - 8N(1-\epsilon)(1-\lambda)}}{4(1-\epsilon)}. \tag{11}$$

The finiteness of the flow requires $\alpha_-(\lambda) = \frac{1}{2}$ or $\alpha_-(\lambda) \geqslant 1$; then we obtain the possible eigenvalues $\epsilon\lambda$ given by

$$\lambda = -\frac{1}{2} + \frac{9-9\epsilon}{2N} + \cdots, \tag{12}$$

$$\lambda \leqslant -1 + \frac{8-8\epsilon}{N} + \cdots. \tag{13}$$

Next we consider the case $v(1) \neq 0$. The existence of the solution given in the expansion requires us to vanish the coefficient of $v(1)$, which determines λ:

$$\lambda = \lambda_1 = 1 + \frac{\epsilon}{N} + \cdots. \tag{14}$$

The corresponding special solution is represented in terms of the generalized hypergeometric function

$$v_1(z) = \frac{\epsilon}{2-2\epsilon} + \frac{2-3\epsilon}{2-2\epsilon}\,{}_3F_2\left(1, 1, 2, ; 1-\alpha_+(\lambda_1), 1-\alpha_-(\lambda_1); \frac{1-z}{2}\right). \tag{15}$$

The finite solution is given by a linear combination of this special solution $v_1(z)$ and a solution of the homogeneous equation with $v(1) = 0$ and with λ given in equation (14)

$$v(z) = a_0^+ v(z, \alpha_+(\lambda_1)) + v_1(z). \tag{16}$$

Since the coefficient a_0^+ should be chosen for the cancellation of the singularity at $z = -1$, this relevant mode is unique. Therefore, the fixed point is singly unstable; then, dimensional reduction can be observed in the critical exponents η and $\bar{\eta}$ for $N \geqslant 18 - \frac{49}{5}\epsilon$. This result agrees with a nonperturbative renormalization group obtained by Tarjus and Tissier [8] and a simple $1/N$ expansion [10]. All eigenfunctions are singular at $z = 1$. It is interesting that the eigenfunction (16) has an essential singularity at $1/N = 0$. This is because of the fact that we solve the eigenvalue equation nonperturbatively without a $1/N$ expansion after the derivation of the eigenvalue equation. Note that the two-loop renormalization group is useful for showing the existence of the relevant mode. The limit $\epsilon \to 0$ of the relevant mode (15) corresponds to the eigenfunction of the one-loop scaling operator with the subleading correction in the $1/N$ expansion. The series expansion for the relevant mode seems to be ill-defined, since

$$\lim_{\epsilon\to 0}[1 - \alpha_+(\lambda_1)] = 2 - \frac{N}{2}$$

is a negative integer for even $N \geqslant 6$. This apparent ill-definition disappears by the two-loop or other higher order corrections. Higher order calculation is provided in our recent work [14].

Here, we comment on the infinitely many relevant modes pointed out by Fisher [5]. To compare our result to Fisher's one-loop renormalization group analysis, we take the limit $\epsilon \to 0$ in the solution (11). They are included in the following series:

$$\alpha_+(\lambda_k) = k - 1, \qquad (k = 3, 4, 5, \ldots) \qquad \text{and} \qquad a_0^- = 0.$$

These belong to the eigenvalues $\epsilon\lambda_k$ given by

$$\lambda_k = 1 - k + \frac{2k^2}{N} + \mathrm{O}\left(\frac{1}{N^2}\right),$$

which are positive for sufficiently large k. These agree with the eigenvalues obtained by Fisher in [5]. Since these relevant modes diverge at $z = -1$, we have eliminated them as unphysical modes. Therefore, perturbation by these infinitely many relevant modes does not exist in a realistic model. On the basis of Fisher's argument for the infinitely many relevant modes, Tissier and Tarjus discussed that the fixed point should have a weak singularity [15].

We consider a possible non-analytic deformation of the fixed point corresponding to the dimensional reduction $R_{\mathrm{DR}}(z)$ in the following form:

$$\begin{aligned} R(z) = R_{\mathrm{DR}}(z) &+ a(1-z)^p + a'(1-z)^{p+1} + a''(1-z)^{p+2} + \cdots \\ &+ b(1-z)^{2p-2} + b'(1-z)^{2p-1} + b''(1-z)^{2p} + \cdots \\ &+ \cdots . \end{aligned} \tag{17}$$

In the one-loop order, the fixed point $R_{\mathrm{DR}}(z)$ can be expanded in $1-z$

$$R'_{\mathrm{DR}}(z) = \frac{\epsilon}{N-2} + \epsilon \frac{N-8-\sqrt{(N-2)(N-18)}}{2(N-2)(N+7)}(z-1) + \cdots .$$

The fixed-point condition $\beta_0[R] + \beta_1[R] = 0$ in the one-loop beta function gives the following condition:

$$\begin{aligned} 2(N+7)[2p^2 - (N-1)p + N - 2] &+ p(N-5+6p) \\ &\times \left[N - 8 - \sqrt{(N-2)(N-18)}\right] = 0, \end{aligned}$$

which determines p. Other coefficients $a', a'' \cdots, b, \cdots$ are determined recursively as functions of a. Since there is no condition on the parameter a, this deformation leads to a fixed line. This fixed line corresponds to deformation by the eigenfunction with $\lambda = 0$ in our linear analysis. In the $1/N$ expansion we have $p = \alpha + 1 = \frac{N}{2} - \frac{7}{2} + \cdots$, which agrees with the higher order calculation in the $1/N$ expansion obtained in [14]. In our previous argument, however, this eigenfunction is unphysical, since it diverges at $z = -1$ with $(1+z)^{-\frac{N}{2}}$. Therefore, this fixed line is unphysical, and the fixed point $R_{\mathrm{DR}}(z)$ is the unique singly unstable fixed point.

In this paper, we have studied the stability of the fixed point corresponding to dimensional reduction on the basis of the two-loop renormalization group obtained by Le Doussal and Wiese [12] and Tissier and Tarjus [13]. The fixed point corresponding to dimensional reduction cannot exist for $N < 18 - \frac{49}{5}\epsilon$. This condition on (d, N) agrees with the phase diagram obtained by Tarjus and Tissier [8] in a nonperturbative renormalization group and also with an existence condition for the cuspy fixed point obtained by the Le Doussal and Wiese [12]. We derive the eigenvalue equation of the scaling operator in the double expansion in $1/N$ and ϵ on the basis of the two-loop renormalization group, and solve it nonperturbatively. We show that the unique singly unstable fixed point gives the critical exponents η and $\bar{\eta}$ predicted by dimensional reduction. This result agrees with that obtained by Tarjus and Tissier in the nonperturbative renormalization group [8], and also with the stability of the replica-symmetric saddle-point solution in the $1/N$ expansion [10].

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